Dynamical role of anyonic excitation statistics in rapidly rotating Bose gases

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We show that for rotating harmonically trapped Bose gases in a fractional quantum Hall state, the anyonic excitation statistics in the rotating gas can effectively play a *dynamical* role. For particular values of the two-dimensional coupling constant $g = -2\pi\hbar^2(2k-1)/m$, where k is a positive integer, the system becomes a noninteracting gas of anyons, with exactly obtainable solutions satisfying Bogomol'nyi self-dual order parameter equations. Attractive Bose gases under rapid rotation thus can be stabilized in the thermodynamic limit due to the anyonic statistics of their quasiparticle excitations.

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Introduction.–Classical and quantum properties of atomic Bose gases when they are set under rotation were intensely studied in the last couple of years both from the experimental [1, 2, 3] and the theoretical sides [4, 5, 6, 7, 8, 9, 10, 11]. Of particular interest is the behavior of rapidly rotating two-dimensional (2D) gases in a fractional quantum Hall state, corresponding to an electrically neutral, bosonic analog of ultrapure 2D electron gases in very strong magnetic fields, which exhibits strongly correlated physics [4, 5]. It is well established that fractional quantum Hall states possess anyonic excitations above an incompressible ground state [12], carrying certain fractions ν of the "elementary" charge [13], where $\nu < 1$ is the filling factor of the lowest Landau level. On the low-energy level of an effective field theory, these systems are described by their coupling to a fictitious statistical gauge field $\mathcal{A}^\mu$. This is known as a Chern-Simons effective field theory description of the fractional quantum Hall effect [14, 15].

In the thermodynamic limit, a system of untrapped bosons with attractive interaction is unstable against collapse and can stably exist only for a finite trapped number of particles [16, 17]. The purpose of the present contribution is to point out that a 2D harmonically trapped rotating Bose gas with attractive interactions can be stabilized by the statistical gauge field associated with the fractional charge of its quasiparticle excitations. The anyonic nature of the excitations can therefore play a *dynamical* role, in that it can compensate the negative coupling constant associated to the interaction of atomic bosons. This dynamical role is due to the fact that the statistical "magnetic" field is proportional to the density of the system, resulting in particle-statistical flux composites. For a special value of the statistics parameter $\Theta = \nu/[2\pi(1-\nu)]$ or, correspondingly, for given Θ, at a particular value of the coupling constant $g = -2\pi\hbar^2(\nu^{-1}-1)/m$, the interaction can effectively be eliminated entirely and make the system behave as an *interaction-free* gas of anyons. In physical terms, the stabilization of the attractively interacting gas in the fractional quantum Hall state against the collapse to a singular distribution may be understood by the existence of a 2D analog of Fermi pressure, "anyonic" pressure.

The order parameter distribution for the free anyon gas can be found analytically provided the Bogomol'nyi self-duality identities are satisfied [18], which then leads to topological vortex solutions for the Ginzburg-Landau order parameter [19, 20, 21, 22, 23, 24]. Rotating atomic Bose gases offer a unique opportunity to actually prepare and detect them rather directly and with great accuracy, and to study their *breather soliton* dynamics [20, 21]. Starting from the statistical mechanics of particles in the plane, interacting by contact potentials, other investigations on obtaining a free anyon gas can be found in [25]; however, only repulsive interactions were treated, without externally imposed magnetic or rotation fields.

Abelian statistical gauge field.–We assume that we have a 2D gas in the fractional quantum Hall regime which admits a Ginzburg-Landau type description in terms of a Chern-Simons theory [15]. A statistical (2+1)D gauge three-potential $\mathcal{A}^\mu$ may be implemented by showing the physical equivalence of the two Hamiltonian theories with $\mathcal{A}^\mu = 0$ and $\mathcal{A}^\mu \neq 0$ on the quantum level [14]. The primary effect of the statistical vector potential is that it allows for an additional Chern-Simons term in the low-energy action ($\hbar \equiv 1$):

$$S_\Theta = \int d^2\boldsymbol{x}dt \left(i\Psi^*(\partial_t + i\mathcal{A}_0)\Psi - \frac{1}{2m}|\boldsymbol{D}\Psi|^2 - V(\boldsymbol{x})|\Psi|^2 - \mathcal{U}(|\Psi|^2) + \frac{\Theta}{4}\epsilon^{\alpha\beta\gamma}\mathcal{A}_\alpha\mathcal{F}_{\beta\gamma}\right) \quad (1)$$

Here, $\boldsymbol{D}$ is the gauge covariant spatial derivative defined in Eq. (2), the Abelian field strength of the statistical gauge field reads $\mathcal{F}_{\alpha\beta} = \partial_\alpha\mathcal{A}_\beta - \partial_\beta\mathcal{A}_\alpha$, and $\mathcal{U}(|\Psi|^2)$ is the interaction energy density. We use relativistic notation, i.e., $\alpha, \beta, \gamma, \mu = 0, 1, 2$; the spacetime metric is diag $(1, -1, -1)$ and repeated indices are summed over. All higher derivative terms in the statistical gauge potentials (e.g., Maxwell terms $\propto \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$) are left out in this low-energy, low-momentum expression [26].

Classically, the momenta of the particles making up the system are given by the decomposition $\boldsymbol{p} = m\boldsymbol{v} + \kappa\boldsymbol{A} + \mathcal{A}$,

(the spatial part of $\mathcal{A}^\mu$ is denoted $\mathcal{A}$), so that the gauge covariant derivative in the quantum domain is given by

$$\boldsymbol{D} = \nabla - i\kappa \boldsymbol{A} - i\mathcal{A}. \tag{2}$$

Here, $\boldsymbol{A}$ is the external U(1) vector potential of rotation, yielding the Coriolis force, with $\nabla \times \boldsymbol{A} = 2\Omega$ twice the applied rotation field Ω perpendicular to the 2D plane, and $\kappa = m$ is the coupling constant for a rotating Bose gas. The total scalar potential

$$V = V_{\rm trap} - \frac{1}{2} m\Omega^2 r^2 \tag{3}$$

consists of (harmonic) trapping potential and centrifugal potential. Due to the Chern-Simons term, the field strength of the statistical gauge field is given by the current as follows:

$$\frac{\Theta}{2}\epsilon^{\mu\alpha\beta}\mathcal{F}_{\alpha\beta} = J^\mu. \tag{4}$$

The homogeneous Maxwell equation for the statistical field strength, $\partial_\mu \mathcal{F}_{\alpha\beta} + \partial_\beta \mathcal{F}_{\mu\alpha} + \partial_\alpha \mathcal{F}_{\beta\mu} = 0$, is then automatically equivalent to current conservation, $\partial_\mu J^\mu = 0$.

The statistical gauge field strength is according to (4) *dual* to the current. This means, in particular, that the statistical magnetic field is proportional to the density:

$$-\Theta\mathcal{B} = |\Psi|^2 = \rho. \tag{5}$$

That is, the order parameter modulus is inextricably linked to the statistical flux, and particle-flux composites are formed. Taking $\Theta > 0$, by fixing $\Omega > 1$, the statistical magnetic field cancels part of the applied "magnetic", i.e., rotation field in the canonical momentum $\boldsymbol{p}$, and the vector potential effectively acting on the particles is reduced. Defining the Landau level filling factor of the original bosonic particles to be $\nu = \pi\rho_0/m\Omega$ [4, 11], with $\rho_0 = |\Psi_0|^2$ a homogeneous background density, we have $\Theta = \nu/[2\pi(1-\nu)]$. In particular, for the $\nu = 1/2$ anyons discussed in [5], $\Theta = 1/2\pi$. The effective magnetic flux of particle-statistical flux composites $\tilde{\Phi} = \oint d\boldsymbol{x}\cdot[\boldsymbol{A} + \mathcal{A}/m] = \nu\Phi_0$, then is reduced compared to the bare rotational flux quantum $\Phi_0 = \oint d\boldsymbol{x}\cdot\boldsymbol{A} = 2\pi/m$ obtained for vanishing $\mathcal{A}$ [27].

The Ginzburg-Landau energy functional of the gas in the rotating frame, corresponding to (1), is composed as usual of kinetic, scalar potential and interaction energy,

$$H = \int d^2\boldsymbol{x} \left(\frac{1}{2m} |\boldsymbol{D}\Psi|^2 + V(\boldsymbol{x})|\Psi|^2 + \mathcal{U}(|\Psi|^2) \right). \tag{6}$$

In the rapid rotation limit $\Omega \simeq \omega_\perp$, $V_{\rm trap} = \frac{1}{2}m\omega_\perp^2 r^2$ is (nearly) cancelled by the centrifugal potential, $V \simeq 0$, where $\omega_\perp$ is the trapping frequency perpendicular to the axis of rotation.

Using the Bogomol'nyi decomposition [18] for the kinetic energy term in the integral (6),

$$|\boldsymbol{D}\Psi|^2 = |(D_1 \pm iD_2)\Psi|^2 \pm \nabla\times\boldsymbol{J} \pm \mathcal{B}\rho \pm 2m\Omega\rho, \tag{7}$$

and relation (5), we can rewrite the Hamiltonian in two different ways corresponding to the $\pm$ sign in (7), to read

$$\begin{aligned} H \;=\; & \int d^2\boldsymbol{x} \left[\frac{1}{2m}|(D_1 \pm iD_2)\Psi|^2 + V(\boldsymbol{x})|\Psi|^2 \right. \\ & \left. + \mathcal{U}(|\Psi|^2) \mp \frac{1}{2}\left(\frac{1}{m\Theta}|\Psi|^2 - 2\Omega\right)|\Psi|^2 \right] \end{aligned} \tag{8}$$

We have neglected the term involving the curl of the matter current $\pm\nabla\times\boldsymbol{J}$, as it amounts, after integration, to a surface contribution vanishing for sufficiently well-behaved current fields and/or well outside the boundary of the rotating gas. We infer from the last term in the second line of (8) that due to the proportionality of density and statistical magnetic field expressed by Eq. (5), and employing the identity (7), part of the kinetic energy $|\boldsymbol{D}\Psi|^2/2m$ can effectively act as interaction energy,

The potential in the Ginzburg-Landau energy functional may generally be expanded

$$\mathcal{U}(|\Psi|^2) = \frac{g}{2}|\Psi|^4 + \frac{\gamma}{6}|\Psi|^6 + \cdots, \tag{9}$$

where g is the value of the two-body coupling constant in the fractional quantum Hall state; for sixth order stability of the system, $\gamma > 0$ is required. Taking into account the coefficient of the quartic term in (8) resulting from the above expansion and Eqs. (4) and (7), the effective interaction coupling of the system is now defined as

$$g_{\rm eff} = g \mp \frac{1}{m\Theta}. \tag{10}$$

If we choose the relation of coupling constant and statistics parameter to be $g = \pm 1/m\Theta$, we see that we have effectively eliminated the quartic interaction coupling term. By "interaction-free" we mean that we have cancelled the interaction g by a proper choice of the statistical magnetic field $\mathcal{B}$, while the covariant derivatives D_1 and D_2 do of course still implicitly contain the statistical gauge fields $\mathcal{A}_1$ and $\mathcal{A}_2$, and therefore, due to (4), the density distribution.

We conclude from relation (10) that a large original coupling g can be converted to a $g_{\rm eff}$ approaching zero. Due to the statistical interaction, a negative g system, unstable towards collapse in the thermodynamic limit [16], may be stabilized by the statistical interaction, in the sense that there are nonsingular distributions $\Psi(\boldsymbol{x}, t)$ (see below), with finite energy. The value $g = g_c \equiv -1/m\Theta$ defines a critical negative coupling strength, below which the gas cannot be stabilized for a given Θ.

In the limit that the coefficient of $|\Psi|^4$ vanishes, $g_{\rm eff} = 0$ in (10), the problem of determining the ground state of zero energy becomes exactly solvable (neglecting the small $|\Psi|^6$ term in the Ginzburg-Landau expansion). The energy per particle then assumes its lower limit $H_0/N =$

$\pm\Omega$, provided the *self-duality* (Bogomol'nyi) constraints [18]

$$(D_1 \pm iD_2)\Psi = 0 \tag{11}$$

are satisfied. To protect the self-dual states of effectively zero coupling, there is an energy barrier to neighboring states in Ψ space, which do not have the property that $g_{\rm eff} = 0$. The magnitude of this energy barrier depends on the (positive) contribution of the $|\Psi|^6$ term in the Ginzburg-Landau type expansion of $\mathcal{U}(|\Psi|^2)$.

Jackiw and Pi [19] have shown that solving the self-duality equations (11) above, in the case of zero "external" field, i.e., in the problem with just the statistical gauge field present and trapping and rotation turned off, $V = 0$ and $\Omega = 0$, is equivalent to solving the Liouville equation

$$\Delta \ln \rho = \pm 2\rho/\Theta \qquad (V(\boldsymbol{x}) = \Omega = 0), \tag{12}$$

whose complete set of solutions is known [28]. To obtain regular and non-negative solutions of the Liouville equation, the sign of the right-hand side must be chosen opposite to that of Θ. Thus the lower, minus sign is appropriate for $\Theta > 0$. The general solution of the Liouville Eq. (12) in the plane of complex $z = x + iy$ reads $\rho(z) = 4\Theta|f'(z)|^2/[1 + |f(z)|^2]^2$, where $f(z)$ is an arbitrary holomorphic function. Radially symmetric vortex solutions, for which $\Psi = \sqrt{\rho}\exp[in\phi]$, take the form, choosing $f(z) = (z_0/z)^n$,

$$\rho(r) = \frac{4\Theta n^2}{r^2}\left[\left(\frac{r_0}{r}\right)^n + \left(\frac{r}{r_0}\right)^n\right]^{-2}, \tag{13}$$

where n is an integer, and r_0 an arbitrary length scale reflecting the scale (dilation) invariance of (12); for $r \to 0$, $\rho(r) \propto r^{2(n-1)}$, and for $r \to \infty$, $\rho(r) \propto r^{-2(n+1)}$.

The solutions of (11) with a constant external magnetic/rotation field (corresponding to our being in the rotating frame), and a linearly increasing electric field in the plane (corresponding to our linear trapping and centrifugal forces) can be obtained from the vortex solitons of Jackiw and Pi: Adding these additional external fields the problem (still) is quadratic [20, 21, 22]. The scalar potential in the Hamiltonian (8), i.e. the effective harmonic trapping field, is very small close to criticality, $V = \frac{1}{2}m\omega_\perp^2 r^2 - \frac{1}{2}m\Omega^2 r^2 \simeq 0$, and can be neglected. The classical problem of finding the solution of (11) in an external rotation field then corresponds to the problem of finding the (semiclassical) Landau levels of anyons [20].

To generate time-dependent vortex soliton solutions, one constructs a coordinate transformation, whose inverse effectively removes the external field and thus leads us back to (13) [20, 21, 22]. The most important feature, apart from the cyclotron motion executed by the soliton, which results from this transformation, is that the size of the soliton *breathes* if the background $\Psi_0 \neq 0$: The scale factor in (13) becomes time dependent according to $r_0 \to r_0 \cos[\Omega t]$. Hence the soliton size oscillates with the applied rotation frequency. Furthermore, the energy of the soliton diverges for $n = 1$ and finite r_0, and thus according to (13) only $n \geq 2$ configurations with $\rho = 0$ at the vortex line can be generated; note that for $n = 1$ the density (13) at the origin has a finite value. The statistical vector potential $\mathcal{A}$ decreases at large distances from the center of the vortex line like $1/r$, as required for a topological vortex of a given quantized circulation. It should also be stressed that the feature that there exists a breather soliton solution in the presence of the external rotation field, persists when $g \neq g_c$ and there is no self-duality fulfilled according to (11); however, in that case simple analytical solutions like the one displayed in (13) cannot be obtained, because the Ginzburg-Landau equations then remain essentially nonlinear.

Rescaling the coordinate vector via $\boldsymbol{x} = \tilde{\boldsymbol{x}}\sqrt{\Theta/2\rho_0}$, the typical length scale of inhomogeneous solutions of the equations (11) is set by $\xi_0 = \sqrt{\Theta/2\rho_0} = \sqrt{1/2m\rho_0|g|}$, the analog of the coherence length in the repulsive case. The topological, i.e., quantized circulation vortex solutions in a rapidly rotating gas with $g < 0$, resulting from the boosted solutions (13), then are cousins of their counterparts in the repulsive-interaction superfluid. In the latter case, the core size of the vortices depends on the applied rotation rate [9] (for an experimental verification see the second reference of [2]). Here, by contrast, the scale r_0 in Eq. (13) is essentially a free parameter, because the total energy of the soliton is minimized for $r_0 = 0$ [21]. Therefore, an external force, created by a blue-detuned laser for example, has to be applied to the gas to generate a soliton with a finite value of r_0. It should be noted that $\Omega \neq 0$ is strictly necessary to obtain any solution of nonzero Ψ_0, i.e., the symmetry breaking expressed by $\Psi_0 \neq 0$ is *induced* by the rotation of the gas: The solutions of (12), e.g., the vortex solution in Eq. (13), are all asymptotically vanishing, $\Psi_0 = 0$ if $\Omega = 0$.

Non-Abelian case.–The previous considerations can be generalized to the case of a noncommuting statistical gauge field (cf., e.g., [23, 24]), which is of potential relevance in rapidly rotating spinor gases. An outline of such a non-Abelian generalization reads as follows. The non-Abelian field strength is

$$\mathcal{F}_{\alpha\beta} = \partial_\alpha \mathcal{A}_\beta - \partial_\beta \mathcal{A}_\alpha + [\mathcal{A}_\alpha, \mathcal{A}_\beta] \tag{14}$$

where the noncommuting gauge field reads $\mathcal{A}_\mu = \mathcal{A}_\mu^a T^a$, using the Lie algebra of the anti-Hermitian generators T^a of the non-Abelian group, $[T^a, T^b] = f_{abc}T^c$ (summation over a, b, c is implied; f_{abc} are the structure constants of the Lie algebra). The Chern-Simons-Gauss law (5) reads in its non-Abelian form

$$\mathcal{B} = \frac{i}{\Theta}T^a(\boldsymbol{\psi}^\dagger T^a \boldsymbol{\psi}), \tag{15}$$

where $\boldsymbol{\psi}$ is the order parameter field, transforming according to some given representation of the gauge group.

The scalar and vector interactions in a spinor condensate may be parametrized to read [29, 30]

$$V_{\rm int} = \frac{c_0}{2}\psi_i^\dagger \psi_j^\dagger \psi_j \psi_i - \frac{c_2}{2}\psi_i^\dagger \psi_k^\dagger T^a_{ij} T^a_{kl} \psi_l \psi_j, \tag{16}$$

where the summation indices i, j, k, l cover the matrix index of the particular representation of $\boldsymbol{\psi}$ chosen. We infer, using (15), that the effective coefficient of the quartic spin-spin interaction term may in analogy to (10) be defined, restricting ourselves to the lower sign,

$$(c_2)_{\rm eff} = c_2 + \frac{1}{m\Theta}. \tag{17}$$

The statistical interaction therefore is able to change, assuming $c_0 < 0$, the spin-spin interaction from $c_2 < 0$ (ferromagnetic) to $c_2 > 0$ ("polar" [29]). However, for a two-parameter interaction Hamiltonian like (16), with $c_0 \neq 0$, there is no self-dual Bogomol'nyi point in parameter space. Only at $c_0 = 0$, such a point exists; then, the system, in general, can support vortices obeying non-Abelian fractional statistics. The matter density components associated to these fractional vortices are solutions of the Toda equation, which generalizes the Liouville Eq. (12) [23, 24].

Conclusion.–We have established the fact that a rapidly rotating Bose gas with attractive self-interaction can be made manifestly stable in a given fractional quantum Hall state, such that for certain values of the negative coupling constant the Ginzburg-Landau matter wave field exhibits self-duality. The resulting free anyon gas interpolates between the extreme cases of bosons (zero statistical pressure at $T = 0$) and fermions (maximal statistical pressure at $T = 0$), and the original bosonic gas is protected against collapse because of this effective "anyonic" pressure. The preparation and experimental verification of this particular fractional quantum Hall state should, in principle, be possible starting from a rotating Bose gas with small positive interaction coupling and high angular momentum $L = N(N-1)/2\nu = \mathcal{O}(N^2)$. To avoid the problem of stabilizing a bulk gas at these very large angular momenta, one can conceive of putting the system on an optical lattice [31]. Starting from the small positive interaction coupling Bose gas, one switches over a close-lying Feshbach resonance to the negative coupling strength side of the resonance. At the end point of the sweep a coupling is chosen which fulfills $g = -2\pi(2k-1)/m$, corresponding to a fractional quantum Hall state at filling $\nu = 1/2k$.

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